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On primitive ideals in polynomial rings over nil rings

Agata Smoktunowicz

Institute of Mathematics, Polish Academy of Sciences, 00-956 Warsaw 10, Śniadeckich 8, P.O.Box 21, Poland.

Abstract

Let R be a nil ring. We prove that primitive ideals in the polynomial ring $R[x]$ in one indeterminate over R are of the form $I[x]$ for some ideals I of R.

All considered rings are associative but not necessarily have identities. Köthe's conjecture states that a ring without nil ideals has no one-sided nil ideals. It is equivalent [4] to the assertion that polynomial rings over nil rings are Jacobson radical. Our main result states that if R is a nil ring and I an ideal in $R[x]$ (the polynomial ring in one indeterminate over R) then $R[x]/I$ is Jacobson radical if and only if $R/I'[x]$ is Jacobson radical, where I' is the ideal of R generated by coefficients of polynomials from I. Also if R is a nil ring and I is a primitive ideal of $R[x]$ then $I = M[x]$ for some ideal M of R. It was asked by Beidar, Fong and Puczyłowski [1] whether polynomial rings over nil rings are not (right) primitive. We show that affirmative answer to this question is equivalent to the Köthe conjecture. We also answer in the negative Question 2 from [1] (Corollary 1).

It is known that if a polynomial ring $R[x]$ is primitive then R need not be primitive [3] (see also Bergman's example in [5]). Let R be a prime ring and I a nonzero ideal of R. Then R is a primitive ring if and only if I is a primitive ring [6]. Since the Hodges example has a nonzero Jacobson radical it follows that polynomial rings over Jacobson radical rings can be right and left primitive (see also Theorem 3).

We recall some definitions after [9] (see also [2], [5]).

A right ideal of a ring R is called *modular* in R if and only if there exists an element $b \in R$ such that $a - ba \in Q$ for every $a \in R$. If Q is a modular maximal right ideal of R then for every $r \notin Q$, $rR + Q = R$.

An ideal P of a ring R is right primitive in R if and only if there exists a modular maximal right ideal Q of R such that P is the maximal ideal contained in Q.

In this paper $R[x]$ denote the polynomial ring in one indeterminate over R. Given polynomial $g \in R[x]$ by $\deg(g)$ we denote the degree of R, i.e., the minimal number d such that $g \in R + Rx + \ldots + Rx^d$. Given $a \in R$ let $< a >_R$ denote the ideal generated by a in R and $< a >_{R[x]}$ the ideal generated by a in $R[x]$. For a ring R, by $J(R)$ we denote the Jacobson radical of R.

We write $I \triangleleft R$ if I is a two-sided ideal of a ring R, and $Q \triangleleft_r R$ if Q is a right ideal of R.

Lemma 1 *Let R be a ring , $r \in R$, $f \in R[x]$, $Q \triangleleft_r R[x]$ and $b \in R[x]$ be such that $a - ba \in Q$ for every $a \in R[x]$. Suppose $b - xrf \in Q$. Then for every i there is $f_i \in R[x]$ such that $b - x^i r f_i \in Q$ and $\deg f_i \leq \deg f$ for all i.*

Proof. We proceed by induction on n. If $n = 1$ we put $f_1 = f$. Suppose Lemma holds for some $n \geq 1$ with $f_n = a_0 + \ldots + a_k x^k$ ($k \leq \deg f$). Then Lemma holds for $f_{n+1} = fra_0 + \sum_{i=1}^{k} a_i x^{i-1}$.

Lemma 2 *Let R be a ring and $I \triangleleft R[x]$ and $a_0 + a_1 x + \ldots + a_k x^k \in I$, $a_0, \ldots, a_k \in R$. Denote $U = < a_k >_{R[x]}$.*

1. *Suppose $h \in U^l$ for some $l \geq 1$ and $\deg h \geq k$. Then there is $g \in U^{l-1}$ such that , $h - g \in I$ and $\deg g < \deg h$.*

2. *Suppose $Q \triangleleft_R R[x]$, $I \subseteq Q$. Let $b \in R[x]$ be such that $a - ba \in Q$ for every $a \in R[x]$. If $b - xrf \in Q$ and $b - g \in Q$ where $f, g \in R[x], g \in U^{\deg f}$ then for every $i > \deg g$ there is $g_i \in R[x]$ such that $b - x^i r g_i \in Q$ and $\deg g_i < k$.*

Proof. 1. Let $h = \sum_{i=0}^{t} c_i x^i$, $t = \deg h$. Since $h \in U^l$ then $c_t = \sum_i p_i a_k q_i$ for some $p_i, q_i \in R \cup \{1\}$, $q_i \in U^{l-1}$. We put $g = h - c_t x^t - \sum_i p_i (\sum_{j=0}^{k-1} a_j x^j) q_i x^{t-k}$.
2. Let f_i be as in Lemma 1 applied for f. Let $g = \sum_{i=0}^{t} c_i x^i$, $c_i \in R$, $c_i \in U^{\deg f}$. For a natural $n > t$, we put $h_n = \sum_{i=0}^{t} f_{n-i} c_i$. Observe that $\deg h_n \leq \deg f$, $h_n \in U^{\deg f}$ and $b - x^n r h_n \in Q$. Applying several times Lemma 2.1 for $h = h_n$, we get that there are $g_n \in R[x]$ such that $b - x^n r g_n \in Q$ for all $n > t$ and $\deg g_n \leq k$. We are done.

Let R be a ring. Given $r \in R$ and $Q \triangleleft_r R$ and $b \in R[x]$ such that $a - ba \in Q$ for every $a \in R[x]$ we say that v is a "good number for r" if for all sufficiently large n, there are $f_n \in R[x]$ such that $\deg f_n \leq v$ and $b - x^n r f_n \in Q$.

Lemma 3 *Let R be a ring, $Q \triangleleft_R R[x]$, and $b \in R[x]$ be such that $a - ba \in Q$ for every $a \in R[x]$. Let $r \in R$, $r \notin Q$ and let v be a good number for all $a \in rR$, $a \notin Q$. Suppose there are $p, p' \notin Q$, $p, p' \in rR$ such that $(pR+Q) \cap (p'R+Q) \subseteq Q$. Then $v - 1$ is a good number for r.*

Proof. Since v is a good number for p, p' then for sufficiently large n there are $g_n \in pR[x]$, $g'_n \in p'R[x]$ with $\deg g_n, \deg g'_n \leq v$ such that $b - x^n g_n \in Q$ and $b - x^n g'_n \in Q$. Let $g_n = p_{n,0} + p_{n,1}x + \ldots + p_{n,v}x^v$, $g'_n = p'_{n,0} + p'_{n,1}x + \ldots + p'_{n,v}x^v$, where all $p_{n,i} \in pR, p'_{n,i} \in p'R$. If for all sufficiently large n either $p_{n,v} \in Q$ or $p'_{n,v} \in Q$ then $v - 1$ is a good number for r. Suppose that there is m such that $p_{m,v} \notin Q$ and $p'_{m,v} \notin Q$. Since $p_{m,v} \in pR$, $p'_{m,v} \in p'R$ then $p_{m,v} - p'_{m,v} \notin Q$ since otherwise $p_{m,v} \in (pR + Q) \cap (p'R + Q)$ contrary our assumptions. Since

v is a good number for $c = p_{m,v} - p'_{m,v}$ then for sufficiently large n there are $h_n \in R[x]$, $\deg h_n \leq v$, $b - x^n c h_n \in Q$. Now $ch_n = c\bar{g}_n + cr_n x^v$ for some $r_n \in R$, $\bar{g}_n \in R[x]$, $\deg \bar{g}_n \leq v-1$. Thus (for sufficiently large n) $b - x^n k_n \in Q$ where $k_n = ch_n + (g'_m - g_m)r_n = c\bar{g}_n + \sum_{i=0}^{v-1}(p'_{m,i} - p_{m,i})x^i r_n$. Note that $k_n \in rR[x]$ since $p, p' \in rR$. Since $\deg k_n \leq v-1$ then $v-1$ is a good number for r.

Theorem 1 *Let R be a nil ring and I be a primitive ideal in $R[x]$ (if any). Then $I = I'[x]$ for some ideal I' in R.*

Proof. Suppose the contrary, that there are $a_0, a_1, \ldots, a_k \in R$, $a_k \notin I$ such that $a_0 + a_1 x + \ldots + a_k x^k \in I$. From the definition there is a modular maximal right ideal Q in $R[x]$ such that I is the maximal possible ideal contained in Q. Now for every $r \notin Q$, $r \in R[x]$, $rR[x] + Q = R[x]$. Since Q is modular, there is $b \in R[x]$ such that $a - ba \in Q$ for every $a \in R[x]$. Observe first that if $r \notin Q$ then $rx \notin Q$ (otherwise $xR \subseteq Q, xR \subseteq I$, impossible since R is nil). Thus there is $f \in R[x]$ such that $b - xrf \in Q$. Denote $U =< a_k >_{R[x]}$. Since I is a primitive ideal it is prime. Consequently $U^{\deg f} \not\subseteq I$, $U^{\deg f} \not\subseteq Q$. Hence $b - g \in Q$ for some $g \in U^{\deg f}$. By Lemma 2.2 we get that $k-1$ is a good number for all $r \in R$, $r \notin Q$. Let v be minimal such that v is a good number for all $r \in R$, $r \notin Q$. We will show that $v = 0$ and hence get a contradiction (since R is a nil ring). Suppose $v > 0$. Let $r \in R$, $r \notin Q$. We will show that $v-1$ is a good number for r. Since v is a good number for r then for some i there are $f_i, f_{i+1}, \ldots, f_{i+k} \in rR[x]$ such that $b - x^j f_j \in Q$ where $i \leq j \leq i+k$. Now each f_j can be written as $f_j = g_j + x^v c_j$, $c_j \in rR$, $g_j \in R[x]$, $\deg g_j < v$. Let $e_j = c_j^{n_j}$ where n_j is minimal possible such that $c_j^{n_j} \notin Q$ where $i \leq j \leq i+k$. (if $c_j \in Q$ we put $e_j = 1$). Now either $v-1$ is a good number for r or there is $s \notin Q$ such that $s \in \bigcap_{j=i}^{i+k}(e_j R + Q)$ and $s \in e_i R$ (by Lemma 3). Then $s - e_j d_j \in Q$ for some $d_j \in R$, $i \leq j \leq i+k$. Since v is a good number for s then for sufficiently large n there are $\bar{f}_n \in sR[x]$, $\deg \bar{f}_n \leq v$ such that $b - x^n \bar{f}_n \in Q$. Let $\bar{f}_n = \sum_{j=0}^{v} s b_j x^j$. Thus $\bar{f}_n - \sum_{j=0}^{v} e_{i+v-j} d_{i+v-j} b_j x^j \in Q$. Now since $b - x^j f_j \in Q$ then $(b - x^j f_j)e_j d_j \in Q$. Thus $\bar{f}_n - \bar{g}_n \in Q$ where $\bar{g}_n = \sum_{j=0}^{v} x^{i+v-j} f_{i+v-j} e_{i+v-j} d_{i+v-j} b_j x^j$. Note that $\bar{g}_n = x^{i+v}\sum_{j=0}^{v} f_{i+v-j} e_{i+v-j} d_{i+v-j} b_j$. Observe that there are $t_l \in rR[x]$, $\deg t_l \leq v-1$ such that $t_l - f_l e_l \in Q$ for $i \leq l \leq i+v$. Denote $h_n = \sum_{j=0}^{v} t_{i+v-j} d_{i+v-j} b_j$. Then $h_n \in rR[x]$ and $\deg h_n \leq v-1$. Since $b - x^n \bar{f}_n \in Q$ then $b - x^n \bar{g}_n \in Q$. Thus $b - x^{i+v+n} h_n \in Q$. Since it holds for all sufficiently large n we get that $v-1$ is a good number for r.

Theorem 2 *Let R be a nil ring and $I \lhd R[x]$ and let $\bar{I} \lhd R$ be the ideal of R generated by coefficients of polynomials from I. Then $R[x]/I$ is Jacobson radical if and only if $R[x]/\bar{I}[x]$ is Jacobson radical.*

Proof. Suppose the contrary, that $R/\bar{I}$ is Jacobson radical and $R[x]/I$ is not Jacobson radical. Then there is a primitive ideal P of $R[x]/I$ such that $P \neq R[x]/I$. Now $P + I$ is a primitive ideal in $R[x]$. Thus $P + I = \bar{P}[x]$ for some ideal $\bar{P}$ in R by Theorem 1. Since $P + I \neq R[x]$ and $\bar{I} \subseteq \bar{P}$ then $R[x]/\bar{I}[x]$ is not Jacobson radical, a contradiction. The other inclusion is clear.

Corollary 1 *If N is a nil ring then the polynomial ring $N[x]$ can not be homomorphically mapped onto a simple primitive ring.*

Krempa [4] showed that the Köthe conjecture is equivalent to the assertion that polynomial rings over nil rings are Jacobson radical. From this and Theorem 1 we get:

Corollary 2 *The Köthe conjecture is equivalent to the statement "polynomial rings (in one indeterminate) over nil rings are not right primitive".*

Simple Jacobson radical but not nil rings were constructed in [7, 8]. (Rings in [7, 8] are not nil since they satisfy the relation x=yxxy).

Theorem 3 *Let R be a simple Jacobson radical ring which is not nil. Then the polynomial ring $R[x]$ in one indeterminate over R is right primitive.*

Proof. Since R is not nil there is b in R such that b is not nilpotent. Let Q be a right ideal in $R[x]$ maximal with the property that $xb \notin Q$ and $r - xbr \subseteq Q$ for every $r \in R[x]$. Then Q is a maximal modular right ideal in $R[x]$. We will show that if I is a two-sided ideal of $R[x]$ and $I \subseteq Q$ then $I = 0$. Let $a_0 + a_1x + \ldots a_nx^n \in I$ where $a_0, \ldots, a_n \in R, a_n \neq 0$. Since R is a simple ring $n > 0$ and there are $b_j, c_j \in R$, $j = 1, 2, \ldots, m$ such that $\sum_{j=1}^{m} b_ja_nc_j = b$. Denote $g = d_0 + \ldots + d_nx^n$, where $d_i = \sum_{j=1}^{m} b_ja_ic_j$ for $0 \leq i \leq n$. Let $d = \sum_{k=1}^{n-1} b^{n-1-k}d_k$. Note that $r - (bx)^kr \in Q$ for every $r \in R[x]$, $k > 0$. Thus $d - \sum_{k=1}^{n} b^{n-1}x^kd_k \in Q$. Since $g \in I$ then $b^{n-1}g \in Q$. Consequently $d + b^nx^n \in Q$ (since $d_n = b$). Thus $dr + r \in Q$ for every $r \in R[x]$. Since $d \in R$ and R is Jacobson radical and $d \in R$ then $d + dr + r = 0$ for some $r \in R$. Thus $d \in Q$, impossible.

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